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VIBRATION THEORY OF NON-HOMOGENEOUS, SPHERICALLY ISOTROPIC PIEZOELASTIC BODIES

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In this paper, three displacement functions are introduced to simplify the basic equations of a radially polarized, spherically isotropic, piezoelectric medium with radial inhomogeneity. For the general non-axisymmetric free vibration problem, it is shown that the controlling equations are finally reduced to an uncoupled second order ordinary differential equation and a coupled system of three second order ordinary differential equations. Solutions to these differential equations are given for the case that material constants are of power functions of the radial co-ordinate. For free vibrations of multilayered piezoelastic spherical shells, it is shown that there are two separated classes of vibrations. The first class is independent of the electric effect and is identically the same as that for pure elasticity, while the second is affected by the electric field. Numerical results are given for the non-axisymmetric free vibration of a single-layered, inhomogeneous piezoelastic spherical shell and effects of some involved parameters are discussed.

1. INTRODUCTION

Piezoelectric materials (PZMs) have been extensively used as transducers and sensors due to the intrinsic direct and converse piezoelectric effects that take place between the electric field and the mechanical deformation. They play key roles as active components in many branches of science and technology such as electronics, infranics, navigation, piezoelectric power supplies, biology and medical ultrasonic imaging applications. More recently, due to emergence of piezoelectric composites, the use of PZMs has gone beyond the traditional application domain of small electric devices or components. Since Bailey and Hubbard's pioneer work [1], PZMs have been employed as integrated structural elements. These adaptive structures are capable of monitoring and adapting to their environment, providing a "smart" response to the external conditions. Readers are referred to a state-of-art survey by Rao and Sunar [2].

For engineering use, piezoelectric components and elements are always fabricated in a plate or shell configuration and undergo both static and dynamic forces. In particular, some of them work in principle according to their dynamic characteristics. Thus, a comprehensive and thorough understanding of dynamic behaviors of piezoelectric plates and shells is required. Relative investigations before 1980 can be found in a review article by Dokmeci [3]. The most recent results on plates and cylindrical shells include the works [4–12], among others. As regards problems related to spherical shells, Kirichok [13] has studied the radial oscillation of a piezoelectric spherical shell coupled with both inner and outer fluid media. Shul'ga *et al.* [14–17] have also investigated the free and forced radial and axisymmetric vibrations of homogeneous piezoceramic hollow spheres. The fact that some

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piezoceramic converters are made in the spherical configuration inspires us to pay more attention here to the general dynamics of piezoelastic spherical bodies. It is also noted that a spherical body is one of the most simplest and common configurations in the so-called solid resonance method [18], which is very popular nowadays for predicting the physical constants, especially of modern advanced materials such as composites, crystals and ceramics.

When dealing with piezoelectric solids, transverse isotropy is of fundamental importance: the most technologically important PZMs are poled ceramics that exhibit transverse isotropy with the unique axis aligned along the poling direction. In spherical co-ordinates, transverse isotropy is also known to be spherical isotropy with the unique axis along the radial direction. For spherically isotropic pure elasticity, Hu [19] introduced two displacement potentials to represent the displacement components and then simplified the basic equations of equilibrium; he showed that the general solutions may be found through the use of spherical harmonics. On the basis of Hu's separation method, Chen [20] considered some axisymmetric problems such as a concentrated force in an infinite medium, stress concentration due to a spherical cavity and a steadily rotating shell. The separation method also has been employed by Shul'ga [21] to analyze the general electroelastic oscillations of homogeneous spherical shells.

Piezoelectric crystals besides being direction-oriented could also exhibit inhomogeneity with reference to physical properties. For pure elastic problem, Puro [22] applied the separation method to take account of the effect of the radial inhomogeneity of a spherically isotropic elastic medium. Sarma [23] considered the torsional wave motion of an inhomogeneous piezoelectric cylindrical shell with finite length. Recently, the concept of functionally graded material (FGM) has been introduced to describe a special kind of inhomogeneous material. In FGM, there are at least two material constituents that are combined together according to a specific scheme. The material properties of FGMs always vary along one or more directions continuously. To provide new ideas for the design and optimization of smart structures, it is necessary to study the effect of the material graded property of piezoelectric plates and shells.

Frobenius power-series (FPS) method has been proven to be the most powerful tool to obtain series form solutions to differential equations with singular points. Many well-known functions have been constructed and developed such as Legendre and Bessel functions. It might be due to Minkarah and Hoppmann [24] who analyzed the vibrations of anisotropic circular plates that the FPS method firstly attracted attention from researchers of mechanics. Mirsky almost simultaneously used the FPS method to consider the free vibration of infinite orthotropic cylinders [25] and later of orthotropic cylindrical shells [26]. From then on, the FPS method has been widely used to analyze problems in various aspects of mechanics, including the recent application in studying static behavior of piezoelectric laminated cylindrical shells [27].

The traditional FPS method used to solve differential equations has the drawback that it is difficult to completely establish all relationships between the roots of the characteristic equation (indicial equation). Recently, Ding *et al.* [28, 29] proposed a matrix FPS method to overcome the above-mentioned drawback and solved the related ordinary differential equations appearing in vibrations of spherically isotropic elastic hollow spheres [30].

In the paper, the separation method is further generalized and applied to study the general vibration problem of a spherically isotropic piezoelastic medium with radial inhomogeneity. The basic equations of a spherically isotropic piezoelastic body are briefly reviewed in section 2. Three displacement functions are then introduced to decompose three displacement components in spherical co-ordinates in section 3. Some considerations on the solutions are presented in section 4. It is found that for the usual transversely isotropic homogeneous piezoelectricity, some results degenerate identically to those available in the literature. Section 5 considers the general non-axisymmetric free vibration problem. By expanding three displacement functions and the electric potential in terms of spherical harmonics, the controlling equations are further simplified to an uncoupled second order ordinary differential equation and a coupled system of three such equations. Since any sufficiently continuous function can be expressed in power series by virtue of the Taylor's theorem, attention is paid to the case that the material constants are of power functions in the radial variable. In this case, solutions to the independent equation as well as the coupled system are derived in section 6. The free vibration problem of multi-layered piezoelectric spherical shells is then considered in section 7 with exact frequency equations presented in section 8 for a single-layered spherical shell. Numerical results are then given in section 9 to discuss the effects of some involved parameters.

2. BASIC EQUATIONS

For a spherically isotropic piezoelastic medium, spherical co-ordinates (r, θ, ϕ) are used with r radial, θ colatitudinal and ϕ meridional. Supposing the center of anisotropy to be identical to the origin of the co-ordinates, the linear constitutive relations are expressed as follows [21]:

$$\begin{aligned}
&\sigma_{\theta\theta} = c_{11}s_{\theta\theta} + c_{12}s_{\phi\phi} + c_{13}s_{rr} - e_{31}E_r, \ \sigma_{r\theta} = 2c_{44}s_{r\theta} - e_{15}E_\theta, \\
&\sigma_{\phi\phi} = c_{12}s_{\theta\theta} + c_{11}s_{\phi\phi} + c_{13}s_{rr} - e_{31}E_r, \ \sigma_{r\phi} = 2c_{44}s_{r\phi} - e_{15}E_\phi, \\
&\sigma_{rr} = c_{13}s_{\theta\theta} + c_{13}s_{\phi\phi} + c_{33}s_{rr} - e_{33}E_r, \ \sigma_{\theta\phi} = 2c_{66}s_{\theta\phi},
\end{aligned} \tag{1}$$

$$\begin{aligned}
D_\theta &= 2e_{15}s_{r\theta} + \varepsilon_{11}E_\theta, \\
D_\phi &= 2e_{15}s_{r\phi} + \varepsilon_{11}E_\phi, \\
D_r &= e_{31}s_{\theta\theta} + e_{31}s_{\phi\phi} + e_{33}s_{rr} + \varepsilon_{33}E_r,
\end{aligned} \tag{2}$$

where σ_{ij} and s_{ij} are the stress and strain tensors, respectively, E_i and D_i are the electric field intensity and electric displacement vectors, respectively, c_{ij} are the elastic stiffness constants (measured in a constant electric field), ε_{ij} the dielectric constants (measured at constant strain), and e_{ij} the piezoelectric constants. In the most general case of anisotropy (triclinic crystal structure), the PZM is described by $21 + 6 + 18 = 45$ independent constants. It is noted that for the case of spherical isotropy as represented by equations (1) and (2), we have an additional relationship $c_{11} = c_{12} + 2c_{66}$. Thus, the piezoelectric solid is only characterized by five elastic, two dielectric and three piezoelectric constants, that is, a total of 10 independent material constants. In this paper, we assume that all physical constants including these 10 material ones are functions of the radial co-ordinate r, i.e., the piezoelectric medium under consideration is radially inhomogeneous.

The electric field intensity vector E_i is related to an electric potential Φ as

$$E_r = -\frac{\partial \Phi}{\partial r}, \qquad E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta}, \qquad E_\phi = -\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}. \tag{3}$$

The charge equation of electrostatistics is [21, 31]

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} = \rho_f, \tag{4}$$

where ρ_f is the free charge density.

Apart from equations (1)–(4), the basic equations of a spherically isotropic piezoelastic body still include the strain–mechanical displacement relations and the differential equations of motion. Their forms in spherical co-ordinates can be found in reference [32] and are not repeated here.

3. SEPARATION METHOD AND FORMULATIONS

To simplify the basic equations, three displacement functions w, G and ψ are introduced so that the mechanical displacement components are decomposed as

$$u_\theta = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} - \frac{\partial G}{\partial\theta}, \quad u_\phi = \frac{\partial\psi}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial G}{\partial\phi}, \quad u_r = w. \tag{5}$$

It is further assumed that the body force components F_i ($i = r, \theta, \phi$) can also be decomposed in the same way, i.e.,

$$rF_\theta = -\frac{1}{\sin\theta}\frac{\partial V}{\partial\phi} - \frac{\partial U}{\partial\theta}, \quad rF_\phi = \frac{\partial V}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial U}{\partial\phi}. \tag{6}$$

The most common case is that the body force vector is potential, for which one has

$$V = 0, \qquad F_r = -\frac{\partial U}{\partial r}. \tag{7}$$

By employing equations (5) and (6), through some lengthy manipulations, we can transfer the basic equations to the following equations:

$$\begin{aligned}&\frac{\partial}{\partial\theta}[A + (\nabla_2 c_{44})w - (\nabla_2 c_{44})(\nabla_2 G - G) + (\nabla_2 e_{15})\Phi - rU + r^2\rho\ddot{G}]\\ &\quad - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}[B + (\nabla_2 c_{44})(\nabla_2\psi - \psi) + rV - r^2\rho\ddot{\psi}] = 0,\end{aligned} \tag{8}$$

$$\begin{aligned}&\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}[A + (\nabla_2 c_{44})w - (\nabla_2 c_{44})(\nabla_2 G - G) + (\nabla_2 e_{15})\Phi - rU + r^2\rho\ddot{G}]\\ &\quad + \frac{\partial}{\partial\theta}[B + (\nabla_2 c_{44})(\nabla_2\psi - \psi) + rV - r^2\rho\ddot{\psi}] = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}&[L_3 + 2(\nabla_2 c_{13}) + (\nabla_2 c_{33})\nabla_2]w - [L_4 + (\nabla_2 c_{13})]\nabla_1^2 G\\ &\quad + [L_5 + (\nabla_2 e_{33})\nabla_2]\Phi + r^2 F_r - r^2\rho\ddot{w} = 0,\end{aligned} \tag{10}$$

$$[L_7 + 2(\nabla_2 e_{31}) + (\nabla_2 e_{33})\nabla_2]w - [L_8 + (\nabla_2 e_{31})]\nabla_1^2 G - [L_9 + (\nabla_2 \varepsilon_{33})\nabla_2]\Phi = r^2\rho_f, \tag{11}$$

where ρ is the mass density of the piezoelastic body, which is also a function of r, a dot over any quantity represents its derivative with respect to time t, and

$$A = L_1 w - L_2 G + L_6\Phi, \quad B = [c_{44}\nabla_3^2 - 2c_{44} + c_{11} - c_{12} + \tfrac{1}{2}(c_{11} - c_{12})\nabla_1^2]\psi,$$

$$L_1 = (c_{13} + c_{44})\nabla_2 + c_{11} + 2c_{44} + c_{12}, \quad L_2 = c_{44}\nabla_3^2 - 2c_{44} + c_{11} - c_{12} + c_{11}\nabla_1^2,$$

$$L_3 = c_{33}\nabla_3^2 - 2(c_{11}+c_{12}-c_{13}) + c_{44}\nabla_1^2, \quad L_4 = (c_{13}+c_{44})\nabla_2 - c_{44} - c_{11} - c_{12} + c_{13},$$

$$L_5 = e_{33}\nabla_3^2 - 2e_{31}\nabla_2 + e_{15}\nabla_1^2, \quad L_6 = (e_{15}+e_{31})\nabla_2 + 2e_{15}, \tag{12}$$

$$L_7 = e_{33}\nabla_3^2 + 2e_{31}\nabla_2 + 2e_{31} + e_{15}\nabla_1^2, \quad L_8 = (e_{31}+e_{15})\nabla_2 + e_{31} - e_{15},$$

$$L_9 = \varepsilon_{33}\nabla_3^2 + \varepsilon_{11}\nabla_1^2, \quad \nabla_2 = r\frac{\partial}{\partial r}, \quad \nabla_2^2 = r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}, \quad \nabla_3^2 = \nabla_2^2 + \nabla_2,$$

$$\nabla_1^2 = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

From equations (8) and (9), one obtains

$$A + (\nabla_2 c_{44})w - (\nabla_2 c_{44})(\nabla_2 G - G) + (\nabla_2 e_{15})\Phi - rU + r^2\rho\ddot{G} = \frac{\partial H}{\partial\phi}, \tag{13}$$

$$B + (\nabla_2 c_{44})(\nabla_2\psi - \psi) + rV - r^2\rho\ddot{\psi} = \sin\theta\frac{\partial H}{\partial\theta}. \tag{14}$$

Substituting equations (13) and (14) into equations (8) and (9), one finds

$$\nabla_1^2 H = 0. \tag{15}$$

Hu [19] and Chen [29] have verified that $H \equiv 0$ can be assumed for homogeneous and non-homogeneous spherically isotropic elasticity respectively. A similar demonstration can be given in the case of non-homogeneous piezoelectricity, see Appendix A. Under this situation, equations (13) and (14) become

$$A + (\nabla_2 c_{44})w - (\nabla_2 c_{44})(\nabla_2 G - G) + (\nabla_2 e_{15})\Phi - rU + r^2\rho\ddot{G} = 0, \tag{16}$$

$$B + (\nabla_2 c_{44})(\nabla_2\psi - \psi) + rV - r^2\rho\ddot{\psi} = 0. \tag{17}$$

Thus, we turn the basic equations to equations (10), (11), (16) and (17), from which, we find that the function ψ is uncoupled from two other displacement functions w and G, and the electric potential Φ. In particular, equation (17) is an independent second order partial differential equation in ψ; equations (10), (11) and (16) form a coupled partial differential equation system in w, G and Φ. The separability of the basic equations of non-homogeneous spherically isotropic piezoelectricity will be favorable for solving relative problems.

4. SOME CONSIDERATIONS ON THE SOLUTIONS

4.1. GENERAL CONSIDERATIONS

The solution to equation (17) can be written as

$$\psi = \psi_0 + \psi_1, \tag{18}$$

where ψ_0 is the general solution of the following homogeneous equation:

$$\left[c_{44}\nabla_3^2 - 2c_{44} + c_{11} - c_{12} + \tfrac{1}{2}(c_{11} - c_{12})\nabla_1^2 + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\frac{\partial^2}{\partial t^2}\right]\psi_0 = 0 \quad (19)$$

and ψ_1 is the particular solution of the associated non-homogeneous equation

$$\left[c_{44}\nabla_3^2 - 2c_{44} + c_{11} - c_{12} + \tfrac{1}{2}(c_{11} - c_{12})\nabla_1^2 + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\frac{\partial^2}{\partial t^2}\right]\psi_1 = -rV. \quad (20)$$

The solution to the coupled system can be written as

$$w = \sum_{i=0}^{3} w_i, \qquad G = \sum_{i=0}^{3} G_i, \quad \Phi = \sum_{i=0}^{3} \Phi_i, \quad (21)$$

where w_0, G_0 and Φ_0 are the general solution of the following homogeneous equations:

$$\begin{aligned}
&\left[L_1 + (\nabla_2 c_{44})\right]w_0 - \left[L_2 + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\frac{\partial^2}{\partial t^2}\right]G_0 \\
&\quad + \left[L_6 + (\nabla_2 e_{15})\right]\Phi_0 = 0, \\
&\left[L_3 + 2(\nabla_2 c_{13}) + (\nabla_2 c_{33})\nabla_2 - r^2\rho\frac{\partial^2}{\partial t^2}\right]w_0 - \left[L_4 + (\nabla_2 c_{13})\right]\nabla_1^2 G_0 \qquad (22)\\
&\quad + \left[L_5 + (\nabla_2 e_{33})\nabla_2\right]\Phi_0 = 0, \\
&\left[L_7 + 2(\nabla_2 e_{31}) + (\nabla_2 e_{33})\nabla_2\right]w_0 - \left[L_8 + (\nabla_2 e_{31})\right]\nabla_1^2 G_0 \\
&\quad - \left[L_9 + (\nabla_2 \varepsilon_{33})\nabla_2\right]\Phi_0 = 0
\end{aligned}$$

and w_i, G_i and Φ_i $(i = 1, 2, 3)$ are the particular solutions of the following non-homogeneous equations, respectively.

$$\begin{aligned}
&\left[L_1 + (\nabla_2 c_{44})\right]w_1 - \left[L_2 + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\frac{\partial^2}{\partial t^2}\right]G_1 \\
&\quad + \left[L_6 + (\nabla_2 e_{15})\right]\Phi_1 = rU, \\
&\left[L_3 + 2(\nabla_2 c_{13}) + (\nabla_2 c_{33})\nabla_2 - r^2\rho\frac{\partial^2}{\partial t^2}\right]w_1 - \left[L_4 + (\nabla_2 c_{13})\right]\nabla_1^2 G_1 \qquad (23)\\
&\quad + \left[L_5 + (\nabla_2 e_{33})\nabla_2\right]\Phi_1 = 0, \\
&\left[L_7 + 2(\nabla_2 e_{31}) + (\nabla_2 e_{33})\nabla_2\right]w_1 - \left[L_8 + (\nabla_2 e_{31})\right]\nabla_1^2 G_1 \\
&\quad - \left[L_9 + (\nabla_2 \varepsilon_{33})\nabla_2\right]\Phi_1 = 0
\end{aligned}$$

$$[L_1 + (\nabla_2 c_{44})]w_2 - \left[L_2 + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\frac{\partial^2}{\partial t^2}\right]G_2$$

$$+ [L_6 + (\nabla_2 e_{15})]\Phi_2 = 0,$$

$$\left[L_3 + 2(\nabla_2 c_{13}) + (\nabla_2 c_{33})\nabla_2 - r^2\rho\frac{\partial^2}{\partial t^2}\right]w_2 - [L_4 + (\nabla_2 c_{13})]\,\nabla_1^2 G_2 \quad (24)$$

$$+ [L_5 + (\nabla_2 e_{33})\nabla_2]\Phi_2 = -r^2F_r,$$

$$[L_7 + 2(\nabla_2 e_{31}) + (\nabla_2 e_{33})\nabla_2]w_2 - [L_8 + (\nabla_2 e_{31})]\,\nabla_1^2 G_2$$

$$- [L_9 + (\nabla_2 \varepsilon_{33})\nabla_2]\Phi_2 = 0$$

$$[L_1 + (\nabla_2 c_{44})]w_3 - \left[L_2 + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\frac{\partial^2}{\partial t^2}\right]G_3$$

$$+ [L_6 + (\nabla_2 e_{15})]\Phi_3 = 0,$$

$$\left[L_3 + 2(\nabla_2 c_{13}) + (\nabla_2 c_{33})\nabla_2 - r^2\rho\frac{\partial^2}{\partial t^2}\right]w_3 - [L_4 + (\nabla_2 c_{13})]\,\nabla_1^2 G_3 \quad (25)$$

$$+ [L_5 + (\nabla_2 e_{33})\nabla_2]\Phi_3 = 0,$$

$$[L_7 + 2(\nabla_2 e_{31}) + (\nabla_2 e_{33})\nabla_2]w_3 - [L_8 + (\nabla_2 e_{31})]\,\nabla_1^2 G_3$$

$$- [L_9 + (\nabla_2 \varepsilon_{33})\nabla_2]\Phi_3 = r^2\rho_f.$$

From equations (19), (20), (22)–(25), it can be seen that the following separation method is suitable for obtaining solutions to these equations:

$$T(r, \theta, \phi, t) = T_r(r)\,T_{\theta\phi}(\theta, \phi)\,T_t(t), \quad (26)$$

where $T(r, \theta, \phi, t)$ denotes ψ_i $(i = 0, 1)$, w_j, G_j and Φ_j $(j = 0, 1, 2, 3)$.

The corresponding equations for the static problem can be obtained by eliminating terms involving derivative with respect to time t and all unknowns are independent of t.

4.2. THE HOMOGENEOUS CASE

If the piezoelastic body is homogeneous, then instead of equations (10), (11), (16) and (17), one obtains

$$\left[c_{44}\nabla_3^2 - 2c_{44} + c_{11} - c_{12} + \tfrac{1}{2}(c_{11} - c_{12})\,\nabla_1^2 - r^2\rho\frac{\partial^2}{\partial t^2}\right]\psi = -rV, \quad (27)$$

$$L_1 w - \left(L_2 - r^2\rho\frac{\partial^2}{\partial t^2}\right)G + L_6\Phi = rU,$$

$$\left(L_3 - r^2\rho\frac{\partial^2}{\partial t^2}\right)w - L_4\,\nabla_1^2 G + L_5\Phi = -r^2F_r, \quad (28)$$

$$L_7 w - L_8\,\nabla_1^2 G - L_9\Phi = r^2\rho_f.$$

It is noted here that for static problems in the homogeneous case since the operators in equations (22)–(25) are interchangeable, by employing the operator theory, we can obtain the general as well as the particular solutions to these equations as follows:

$$w_0 = A_{i1}F_0, \qquad G_0 = A_{i2}F_0, \qquad \Phi_0 = A_{i3}F_0 \quad (i = 1, 2, 3), \tag{29}$$

$$w_j = A_{j1}F_j^*, \qquad G_j = A_{j2}F_j^* \qquad \Phi_j = A_{j3}F_j^* \quad (j = 1, 2, 3, \text{ no summation}), \tag{30}$$

where A_{ij} $(i, j = 1, 2, 3)$ are cofactors of the determinant $|D|$. Here D is the following operator matrix:

$$D = \begin{bmatrix} L_1 & -L_2 & L_6 \\ L_3 & -L_4 \nabla_1^2 & L_5 \\ L_7 & -L_8 \nabla_1^2 & -L_9 \end{bmatrix}. \tag{31}$$

F_0 and F_j^* $(j = 1, 2, 3)$ satisfy the following homogeneous and non-homogeneous equations respectively:

$$|D|F_0 = 0, \qquad |D|F_1^* = rU, \qquad |D|F_2^* = -r^2 F_r, \qquad |D|F_3^* = r^2 \rho_f. \tag{32}$$

4.3. TRANSVERSE ISOTROPY

Transverse isotropy is usually described in cylindrical co-ordinates (r_1, ϕ, z) or Cartesian co-ordinates (x, y, z). It can be seen as a limiting case of spherical isotropy through the following limiting procedure:

$$r \sin\theta \to r_1, \quad \cos\theta \to 1, \quad \frac{\partial}{r\,\partial\theta} \to \frac{\partial}{\partial r_1}, \quad \frac{\partial}{\partial r} \to \frac{\partial}{\partial z}, \quad u_\theta \to u_{r_1}, \quad u_r \to u_z. \tag{33}$$

By virtue of equation (33), one has

$$\frac{1}{r^2}\nabla_1^2 \to \frac{\partial^2}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial}{\partial r_1} + \frac{1}{r_1^2}\frac{\partial^2}{\partial \phi^2} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Lambda. \tag{34}$$

Now, assuming $\psi = -\psi^*/r$ and $G = G^*/r$ in equation (5), and utilizing equation (33), one has

$$u_{r_1} = \frac{\partial \psi^*}{r_1 \partial \phi} - \frac{\partial G^*}{\partial r_1}, \quad u_\phi = -\frac{\partial \psi^*}{\partial r_1} - \frac{\partial G^*}{r_1\,\partial \phi}, \quad u_z = w. \tag{35}$$

The above separation formula has been adopted by Ding *et al.* [33] to deal with problems of transversely isotropic piezoelectric media. We also write down equation (6) in this case as

$$F_{r_1} = \frac{\partial V^*}{r_1\,\partial \phi} - \frac{\partial U}{\partial r_1}, \quad F_\phi = -\frac{\partial V^*}{\partial r_1} - \frac{\partial U}{r_1\,\partial \phi}, \tag{36}$$

where $V = -V^*$ is introduced. Thus, equation (17) will take the following form for transverse isotropy:

$$\left[c_{44}\frac{\partial^2}{\partial z^2} + \tfrac{1}{2}(c_{11} - c_{12})\Lambda + \left(\frac{\partial}{\partial z} c_{44}\right)\frac{\partial}{\partial z} - \rho\frac{\partial^2}{\partial t^2}\right]\psi^* + V^* = 0. \tag{37}$$

If the transversely isotropic body under consideration is homogeneous, then equation (37) reduces to equation (19) in Ding *et al.* [33], where the effect of body force was not taken into consideration. By virtue of equations (33)–(35), the coupled system of equations (10), (11) and (16) becomes

$$\begin{bmatrix} (c_{13}+c_{44})\dfrac{\partial}{\partial z}+\dfrac{\partial c_{44}}{\partial z} & -\left[c_{11}\Lambda+c_{44}\dfrac{\partial^2}{\partial z^2}+\dfrac{\partial c_{44}}{\partial z}\dfrac{\partial}{\partial z}-\rho\dfrac{\partial^2}{\partial t^2}\right] & (e_{15}+e_{31})\dfrac{\partial}{\partial z}+\dfrac{\partial e_{15}}{\partial z} \\ c_{44}\Lambda+c_{33}\dfrac{\partial^2}{\partial z^2}+\dfrac{\partial c_{33}}{\partial z}\dfrac{\partial}{\partial z}-\rho\dfrac{\partial^2}{\partial t^2} & -\left[(c_{13}+c_{44})\dfrac{\partial}{\partial z}+\dfrac{\partial c_{13}}{\partial z}\right]\Lambda & e_{15}\Lambda+e_{33}\dfrac{\partial^2}{\partial z^2}+\dfrac{\partial e_{33}}{\partial z}\dfrac{\partial}{\partial z} \\ e_{15}\Lambda+e_{33}\dfrac{\partial^2}{\partial z^2}+\dfrac{\partial e_{33}}{\partial z}\dfrac{\partial}{\partial z} & -\left[(e_{15}+e_{31})\dfrac{\partial}{\partial z}+\dfrac{\partial e_{31}}{\partial z}\right]\Lambda & -\left[\varepsilon_{11}\Lambda+\varepsilon_{33}\dfrac{\partial^2}{\partial z^2}+\dfrac{\partial \varepsilon_{33}}{\partial z}\dfrac{\partial}{\partial z}\right] \end{bmatrix}$$

$$\times\begin{Bmatrix} x \\ G^* \\ \Phi \end{Bmatrix}=\begin{Bmatrix} U \\ -F_z \\ \rho_f \end{Bmatrix}, \tag{38}$$

where the property $F_r \to F_z$ is employed. It can be seen that for a homogeneous piezoelectric body, equation (38) reduces to equation (20) in Ding *et al.* [33] if the right-hand side of equation (38) vanishes. For the homogeneous case, the operators in equation (38) are interchangeable, and the solution to it can be constructed by the operator theory as mentioned above. Readers are also referred to Ding *et al.* [33].

5. GENERAL NON-AXISYMMETRIC FREE VIBRATION

For the free vibration or steady response problem, equations (10), (11), (16) and (17) can be further simplified. We notice that all these equations include the partial operator ∇_1^2, which is defined in equation (12). The following form for the displacement functions and the electric potential is thus taken for a spherical body

$$\begin{aligned} \psi &= \psi_n(r)S_n^m(\theta,\phi)\exp(\mathrm{i}\omega t), \quad w = w_n(r)S_n^m(\theta,\phi)\exp(\mathrm{i}\omega t), \\ G &= G_n(r)S_n^m(\theta,\phi)\exp(\mathrm{i}\omega t), \quad \Phi = \Phi_n(r)S_n^m(\theta,\phi)\exp(\mathrm{i}\omega t), \end{aligned} \tag{39}$$

where $S_n^m(\theta,\phi)=P_n^m(\cos\theta)\exp(\mathrm{i}m\phi)$ are spherical harmonics and $P_n^m(\cos\theta)$ are the associated Legendre functions, n and m are integers, and ω is the circular frequency. For the sake of computational convenience, the following parameters are introduced:

$$\begin{aligned} &U_n(\xi)=\psi_n(r)/R, \quad V_n(\xi)=G_n(r)/R, \quad W_n(\xi)=w_n(r)/R, \\ &X_n(\xi)=\Phi_n(r)\varepsilon_{33}/(e_{33}R), \quad \xi=r/R, \quad \Omega=\omega R/v_2, \\ &f_1=c_{11}/c_{44}, \quad f_2=c_{12}/c_{44}, \quad f_3=c_{13}/c_{44}, \quad f_4=c_{33}/c_{44}, \\ &f_5=e_{15}/e_{33}, \quad f_6=e_{31}/e_{33}, \quad f_7=\varepsilon_{11}/\varepsilon_{33}, \quad f_8=e_{33}^2/(\varepsilon_{33}c_{44}), \end{aligned} \tag{40}$$

where R is a characteristic length, and $v_2=\sqrt{c_4/\rho}$ is the elastic wave velocity.

Substitution of equation (39) into equations (10), (11), (16) and (17), and making use of equation (40), yields

$$\xi^2U_n''+(f_9+2)\xi U_n'+\{\Omega^2\xi^2-[2+(n^2+n-2)(f_1-f_2)/2+f_9]\}U_n=0, \tag{41}$$

$$\xi^2 W_n'' + (f_{10}+2)\xi W_n' + (\Omega^2\xi^2/f_4 + p_1 + 2f_{11})W_n - p_2\xi V_n' \\ - [p_3 - n(n+1)f_{11}]V_n + q_1\xi^2 X_n'' + (q_2 + f_8 f_{12}/f_4)\xi X_n' + q_3 X_n = 0, \tag{42}$$

$$\xi^2 V_n'' + (f_9+2)\xi V_n' + (\Omega^2\xi^2 + p_4 - f_9)V_n - p_5\xi W_n' - (p_6 + f_9)W_n \\ + q_4\xi X_n' + (q_5 + f_{13})X_n = 0, \tag{43}$$

$$\xi^2 X_n'' + (f_{14}+2)\xi X_n' + q_6 X_n - \xi^2 W_n'' - (p_7 + f_{12})\xi W_n' - (p_8 + f_{15})W_n \\ - p_9\xi V_n' - (p_{10} + f_{16})V_n = 0, \tag{44}$$

where a prime denotes differentiation with respect to ξ, and

$$\begin{gathered} p_1 = [2(f_3 - f_1 - f_2) - n(n+1)]/f_4, \quad p_2 = -n(n+1)(f_3+1)/f_4, \\ p_3 = n(n+1)(f_1 + f_2 + 1 - f_3)/f_4, \quad p_4 = f_1 - f_2 - n(n+1)f_1 - 2, \\ p_5 = f_3 + 1, \quad p_6 = f_1 + f_2 + 2, \quad p_7 = 2(f_6+1), \quad p_8 = 2f_6 - n(n+1)f_5, \\ p_9 = n(n+1)(f_5 + f_6), \quad p_{10} = n(n+1)(f_6 - f_5), \quad q_1 = f_8/f_4, \\ q_2 = 2f_8(1-f_6)/f_4, \quad q_3 = -n(n+1)f_5 f_8/f_4, \quad q_4 = -(f_5+f_6)f_8, \\ q_5 = -2f_5 f_8, \quad q_6 = -n(n+1)f_7, \\ f_9 = (\nabla_2 c_{44})/c_{44}, \quad f_{10} = (\nabla_2 c_{33})/c_{33}, \quad f_{11} = (\nabla_2 c_{13})/c_{33}, \\ f_{12} = (\nabla_2 e_{33})/e_{33}, \quad f_{13} = -f_8(\nabla_2 e_{15})/e_{33}, \quad f_{14} = (\nabla_2 \varepsilon_{33})/\varepsilon_{33}, \\ f_{15} = 2(\nabla_2 e_{31})/e_{33}, \quad f_{16} = n(n+1)(\nabla_2 e_{31})/e_{33}. \end{gathered} \tag{45}$$

Notice here that the body forces and the free charge density have been dropped during the derivation of equations (41)–(44). Thus, for free vibration problem, we have turned the original controlling equations to equations (41)–(44) in a non-dimensional form. It can be seen that equation (41) is an independent second order ordinary differential equation in the unknown U_n. Equations (42)–(44) are coupled by the three unknowns V_n, W_n and X_n, and each equation involved is a second order ordinary differential one. It is obvious that $\xi = 0$ is a singular point both of the uncoupled differential equation (41) and the coupled system (42)–(44). If the distributions of the physical constants along the radial direction are known, then one can distinguish which kind of singularity of the point $\xi = 0$ is. In what follows, solutions to equations (41)–(44) will be given for the particular case when all material constants are of power functions in the radial variable.

6. SOLUTIONS TO EQUATIONS (41)–(44)

We assume here that all material constants are of power functions in the non-dimensional radial variable ξ, say, $c_{ij} = c_{ij}^0\xi^\alpha$, $e_{ij} = e_{ij}^0\xi^\alpha$, $\varepsilon_{ij} = \varepsilon_{ij}^0\xi^\alpha$ and $\rho = \rho^0\xi^\alpha$, here c_{ij}^0, e_{ij}^0, ε_{ij}^0 and ρ^0 are constants. Equations (41)–(44) remain unaltered except that the following non-dimensional parameters read as

$$f_9 = f_{10} = f_{12} = f_{14} = \alpha, \quad f_{11} = \alpha f_3/f_4, \quad f_{13} = -\alpha f_5 f_8, \quad f_{15} = 2\alpha f_6, \quad f_{16} = n(n+1)\alpha f_6. \tag{46}$$

It is noted here that the non-dimensional parameters $f_i (i = 1, 2, \dots, 8)$ defined in equation (40) now take forms such as $f_1 = c_{11}^0/c_{44}^0$ and $f_2 = c_{12}^0/c_{44}^0$, etc.

6.1. SOLUTION TO EQUATION (41)

It can be shown that equation (41) is a special case of the confluent hypergeometric differential equation; its solution can be easily obtained as

$$U_n(\xi) = \xi^{-(1+\alpha)/2}[B_{n1}\mathrm{J}_n(\Omega\xi) + B_{n2}\mathrm{Y}_\eta(\Omega\xi)] \quad (n \geqslant 1), \tag{47}$$

where J_η and Y_η are the first and second kinds of Bessel functions, respectively, B_{n1} and B_{n2} are arbitrary constants, and

$$\eta^2 = [(3+\alpha)^2 + 2(n^2 + n - 2)(f_1 - f_2)]/4 > 0. \tag{48}$$

6.2. SOLUTION TO THE COUPLED SYSTEM (42)–(44)

To obtain the solution to this ordinary differential equation system, the matrix FPS method developed in reference [28] is employed. We assume

$$\mathrm{Y}_n = \begin{Bmatrix} W_n \\ V_n \\ X_n \end{Bmatrix} = \sum_{i=0}^{\infty} \begin{Bmatrix} \bar{W}_n \\ \bar{V}_n \\ \bar{X}_n \end{Bmatrix}_i \xi^{-(1+\alpha)/2+s+i} = \sum_{i=0}^{\infty} \bar{Y}_{ni}\xi^{-(1+\alpha)/2+s+i}. \tag{49}$$

Substitution of equation (49) into equations (42)–(44) yields

$$\sum_{i=0}^{\infty} [\mathbf{H}_1\xi^2 + \mathbf{H}_2(s+i)]\bar{Y}_{ni}\xi^{s+i} = 0, \tag{50}$$

where

$$\mathbf{H}_1 = \begin{bmatrix} \Omega^2/f_4 & 0 & 0 \\ 0 & \Omega^2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{H}_2(s+i) = \begin{bmatrix} H_{211} & H_{212} & H_{213} \\ H_{221} & H_{222} & H_{223} \\ H_{231} & H_{232} & H_{233} \end{bmatrix}, \tag{51}$$

$$H_{211} = si^2 - (1+\alpha)^2/4 + p_1 + 2f_{11}, \quad H_{212} = -p_2[si - (1+\alpha)/2] - p_3 + n(n+1)f_{11},$$

$$H_{213} = q_1\{si^2 - (1+\alpha)^2/4 - 2f_6[si - (1+\alpha)/2]\} + q_3,$$

$$H_{221} = -p_5[si-(1+\alpha)/2]-p_6-\alpha, \quad H_{222} = si^2-(1+\alpha)^2/4+p_4-\alpha,$$

$$H_{223} = q_4[si-(1+\alpha)/2]+q_5+f_{13},$$

$$H_{231} = \{si^2-(1+\alpha)^2/4+2f_6[si-(1+\alpha)/2]+p_8+f_{15}\},$$

$$H_{232} = -p_9[si-(1+\alpha)/2]-p_{10}-f_{16}, \quad H_{233} = si^2-(1+\alpha)^2/4+q_6,$$

where the notation $si = s+i$ has been used for compactness. Considering the left-hand side of equation (50), on equating to zero the coefficient of the term of the lowest degree in ξ, one obtains the indicial equation:

$$|\mathbf{H}_2(s)| = 0. \tag{52}$$

This is a sixth order algebraic equation, from which one can solve for six indicials s. Following the derivation as described in references [28, 29], one can find that it is easy to deal with all the cases considering different relationships between the six indicials. Details are, however, omitted and the general solution finally can be expressed as the linear combination of six independent solutions as follows:

$$W_n(\xi) = \sum_{j=1}^{6} C_{nj}W_{nj}(\xi), \quad V_n(\xi) = \sum_{j=1}^{6} C_{nj}V_{nj}(\xi), \quad X_n(\xi) = \sum_{j=1}^{6} C_{nj}X_{nj}(\xi), \tag{53}$$

where C_{nj} are abitrary constants and W_{nj}, V_{nj} and X_{nj} are convergent, infinite series in the variable ξ that can be obtained by comparing the coefficients of equation (50) term by term. It is noted here that for the most common case where no two of the six indicials differ by an even integer, solution (53) has the simplest form containing no logarithmic term.

It should be pointed out that $n=0$ is a special case for which function $V_n(\xi)$ contributes nothing to the piezoelastic field as can one see from equation (5). In fact, equations (42)–(44) will read in this case as follows

$$\begin{aligned}&\xi^2 W_0'' + (2+\alpha)\xi W_0' + (1/f_4)(\Omega^2\xi^2 + 2f_3 - 2f_1 - 2f_2 + 2\alpha f_3)W_0 \\ &\quad + (f_8/f_4)[\xi^2 X_0'' + (2-2f_6+\alpha)\xi X_0'] = 0,\end{aligned} \tag{54}$$

$$\xi^2 X_0'' + (2+\alpha)\xi X_0' - \xi^2 W_0'' - (2f_6+2+\alpha)\xi W_0' - 2f_6(1+\alpha)W_0 = 0. \tag{55}$$

In this case, instead of equation (52), we can derive the following indicial equation:

$$\left[s^2 - \frac{(1+\alpha)^2}{4}\right]\left\{s^2 - \frac{f_4(1+\alpha)^2 - 8(f_3-f_1-f_2+\alpha f_3) + f_8[4f_6-(1+\alpha)]^2}{4(f_4+f_8)}\right\} = 0. \tag{56}$$

We can then use the matrix FPS method to obtain the following form solution of equations (54) and (55):

$$W_0(\xi) = \sum_{i=1}^{4} C_{0i}W_{0i}(\xi), \quad X_0(\xi) = \sum_{t=1}^{4} C_{0i}X_{0i}(\xi). \tag{57}$$

7. FREE VIBRATIONS OF MULTI-LAYERED SPHERICAL SHELLS

Based on equations (1)–(3), (5), (39) and (40), one can write out the expressions for stresses, mechanical and electric displacements, and electric potential on a spherical surface in the non-dimensional functions $U_n(\xi)$, $V_n(\xi)$, $W_n(\xi)$ and $X_n(\xi)$ as follows (for the sake of simplicity, the common dynamic factor $\exp(i\omega t)$ is dropped):

$$u_r = RW_nS_n^m(\theta,\phi), \quad u_\theta = -R\left[\frac{1}{\sin\theta}\frac{\partial S_n^m(\theta,\phi)}{\partial\phi}U_n + \frac{\partial S_n^m(\theta,\phi)}{\partial\theta}V_n\right],$$

$$u_\phi = R\left[\frac{\partial S_n^m(\theta,\phi)}{\partial\theta}U_n - \frac{1}{\sin\theta}\frac{\partial S_n^m(\theta,\phi)}{\partial\phi}V_n\right], \quad \Phi = R\frac{e_{33}}{\varepsilon_{33}}X_nS_n^m(\theta,\phi),$$

$$\sigma_{rr} = c_{13}\,[n(n+1)V_n/\xi + 2W_n/\xi + (f_4/f_3)W_n' + (f_8/f_3)X_n']S_n^m(\theta,\phi), \tag{58}$$

$$\sigma_{r\theta} = c_{44}\left[\frac{\partial S_n^m(\theta,\phi)}{\partial\theta}(W_n/\xi + V_n/\xi - V_n' + f_5f_8X_n/\xi) - \frac{1}{\sin\theta}\frac{\partial S_n^m(\theta,\phi)}{\partial\phi}(U_n' - U_n/\xi)\right],$$

$$\sigma_{r\phi} = c_{44}\left[\frac{1}{\sin\theta}\frac{\partial S_n^m(\theta,\phi)}{\partial\phi}(W_n/\xi + V_n/\xi - V_n' + f_5f_8X_n/\xi) + \frac{\partial S_n^m(\theta,\phi)}{\partial\theta}(U_n' - U_n/\xi)\right],$$

$$D_r = e_{33}[n(n+1)f_6V_n/\xi + 2f_6W_n/\xi + W_n' - X_n']S_n^m(\theta,\phi).$$

Equation (58) is given for a single couple of n and m; the whole electroelastic field is the sum of such terms for n varying from 0 to ∞ and $m \leqslant n$. Because of the orthogonal property of spherical harmonics, we can deal with them separately.

We shall now consider the boundary and the continuity conditions of a multilayered spherical shell composed of p layer piezoelastic materials, as shown in Figure 1. For the free vibration problem, both the inner and outer spherical surfaces are free from tractions as well as the normal electric displacement, i.e.,

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\phi} = 0 = D_r \quad (r = a, b). \tag{59}$$

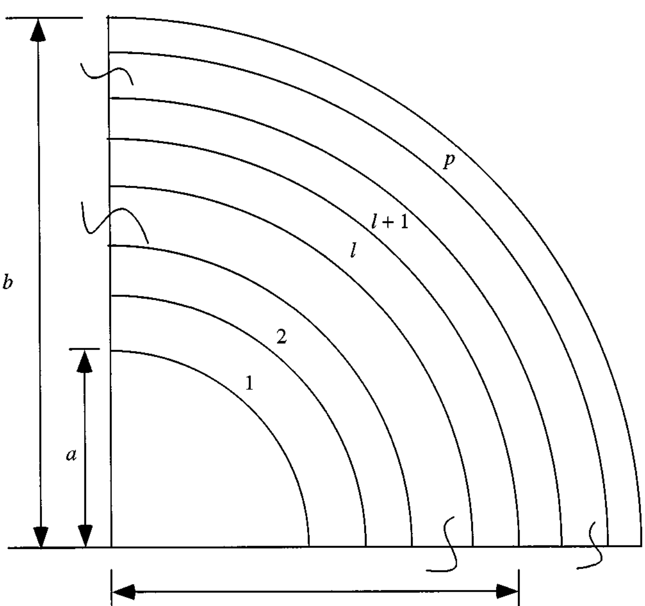

Figure 1. Geometry of a multi-layered spherical shell.

We further suppose that two connected layers say the lth and $(l+1)$th layers, are perfectly bonded so that all the physical quantities at the interface $r = c$ are continuous. Thus, we have

$$\begin{aligned}&\sigma_{rr}^{(l)} = \sigma_{rr}^{(l+1)} \quad \sigma_{r\theta}^{(l)} = \sigma_{r\theta}^{(l+1)}, \quad \sigma_{r\phi}^{(l)} = \sigma_{r\phi}^{(l+1)}, \quad u_r^{(l)} = u_r^{(l+1)} \\ &u_\theta^{(l)} = u_\theta^{(l+1)}, \quad u_\phi^{(l)} = u_\phi^{(l+1)}, \quad D_r^{(l)} = D_r^{(l+1)}, \quad \Phi^{(l)} = \Phi^{(l+1)}\end{aligned} \qquad (r = c), \tag{60}$$

where the superscript (l) denotes the quantity of the lth layer and so on. Noticing the following identity,

$$\sin\theta \frac{\mathrm{d}P_n^m(\cos\theta)}{\mathrm{d}\theta} = \frac{1}{2n+1}[n(n-m+1)P_{n+1}^m(\cos\theta) - (n+1)(n+m)P_{n-1}^m(\cos\theta)] \tag{61}$$

and the orthogonal property of the Legendre functions, one obtains from equations (58) and (59):

$$U_n'^{(1,p)} - U_n^{(1,p)}/\xi = 0 \quad (\xi = \xi_1, \xi_2), \tag{62}$$

$$\begin{aligned}&n(n+1)V_n^{(1,p)}/\xi + 2W_n^{(1,p)}/\xi + (f_4^{(1,p)}/f_3^{(1,p)})W_n'^{(1,p)} + (f_8^{(1,p)}/f_3^{(1,p)})X_n'^{(1,p)} = 0 \\ &W_n^{(1,p)}/\xi + V_n^{(1,p)}/\xi - V_n'^{(1,p)} + f_5^{(1,p)}f_8^{(1,p)}X_n^{(1,p)}/\xi = 0 \qquad (\xi = \xi_1, \xi_2), \\ &n(n+1)f_6^{(1,p)}V_n^{(1,p)}/\xi + 2f_6^{(1,p)}W_n^{(1,p)}/\xi + W_n'^{(1,p)} - X_n'^{(1,p)} = 0\end{aligned} \tag{63}$$

where $\xi_1 = a/R$ and $\xi_2 = b/R$ are the non-dimensional radii and $R = (a+b)/2$ is the mean radius. Similarly, equation (60) gives

$$\begin{aligned}&c_{44}^{(l)}[U_n'^{(l)} - U_n^{(l)}/\xi] = c_{44}^{(l+1)}[U_n'^{(l+1)} - U_n^{(l+1)}/\xi] \\ &U_n^{(l)} = U_n^{(l+1)}\end{aligned} \qquad (\xi = \xi_c = c/R), \tag{64}$$

$$\begin{aligned}&c_{13}^{(l)}[n(n+1)V_n^{(l)}/\xi + 2W_n^{(l)}/\xi + (f_4^{(l)}/f_3^{(l)})W_n'^{(l)} + (f_8^{(l)}/f_3^{(l)})X_n'^{(l)}] \\ &\quad = c_{13}^{(l+1)}[n(n+1)V_n^{(l+1)}/\xi + 2W_n^{(l+1)}/\xi + (f_4^{(l+1)}/f_3^{(l+1)})W_n'^{(l+1)} + (f_8^{(l+1)}/f_3^{(l+1)})X_n'^{(l+1)}] \\ &c_{44}^{(l)}[W_n^{(l)}/\xi + V_n^{(l)}/\xi - V_n'^{(l)} + f_5^{(l)}f_8^{(l)}X_n^{(l)}/\xi] \\ &\quad = c_{44}^{(l+1)}[W_n^{(l+1)}/\xi + V_n^{(l+1)}/\xi - V_n'^{(l+1)} + f_5^{(l+1)}f_8^{(l+1)}X_n^{(l+1)}/\xi] \qquad (\xi = \xi_c = c/R). \\ &e_{33}^{(l)}[n(n+1)f_6^{(l)}V_n^{(l)}/\xi + 2f_6^{(l)}W_n^{(l)}/\xi + W_n'^{(l)} - X_n'^{(l)}] \\ &\quad \sim e_{33}^{(l+1)}[n(n+1)f_6^{(l+1)}V_n^{(l+1)}/\xi + 2f_6^{(l+1)}W_n^{(l+1)}/\xi + W_n'^{(l+1)} - X_n'^{(l+1)}] \\ &W_n^{(l)} = W_n^{(l+1)}, \quad V_n^{(l)} = V_n^{(l+1)}, \quad [e_{33}^{(l)}/\varepsilon_{33}^{(l)}]X_n^{(l)} = [e_{33}^{(l+1)}/\varepsilon_{33}^{(l+1)}]X_n^{(l+1)}\end{aligned} \tag{65}$$

As we can see from equations (62)–(65), *the boundary conditions are also separated into two catalogues*: *one relates to the function* $U_n(\xi)$ only, and the other is expressed by the other three functions $W_n(\xi)$, $V_n(\xi)$ and $X_n(\xi)$. Now we can reach the conclusion that the free vibrations of a multi-layered, inhomogeneous, spherically isotropic, piezoelastic spherical shell can be divided into two independent classes as the cases of a pure elastic shell [29, 34, 35] and of a homogeneous piezoelastic one [21]. The first class is defined by the differential equation (41) and the boundary conditions (62) and (64), while the second by the differential equations (42)–(44) and the boundary conditions (63) and (65). It can be shown that the first class corresponds to an equivoluminal motion of the shell and is characterized by the absence of a radial component of the mechanical displacement and of the electric potential, while, for the second class, the mechanical displacement has in general, both transverse and radial components, but the rotation has no radial component. Substituting solutions of the differential equations (41)–(44) into the boundary and continuity conditions (62)–(65), one can get $2p$ homogeneous linear algebraic equations in $2p$ unknown constants $B_{ni}^{(j)}$ $(i = 1, 2; j = 1, 2, \dots, p)$ for the first class, and $6p$ such ones in $6p$ unknown constants $C_{ni}^{(j)}$ $(i = 1, 2, \dots, 6; j = 1, 2, \dots, p)$ for the second class. It is also noted that $n = 0$ is an exception of the second class, for which, in general, $4p$ homogeneous linear algebraic equations will be obtained; however, it will be shown that only $2p$ ones will be involved finally and a detailed explanation will be given later for the case that material constants obey power laws. It is well known that for non-trivial solutions to exist, the coefficient determinants of the two linear systems should vanish so that the corresponding frequency equations can be obtained. One interesting point that should be mentioned is that the differential equations as well as the boundary conditions, and thus the frequency equations, for both classes, do not contain the integer m, which particularly represents the non-axisymmetric characteristics of the motion of the shell. It seems paradoxical, however, the explanation has already been given by Silbiger [36] for a thin isotropic spherical shell that the non-axisymmetric modes of vibrations can be obtained by the superposition of the axisymmetric ones of identical natural frequency. That is still the case for a multi-layered piezoelastic spherical shell with radial inhomogeneity.

In the following section, we will only give the frequency equations for a single-layered spherical shell as an example; the ones for multi-layered spherical shell can be readily obtained based on the above results. Material constants are assumed to be in power forms along the radial direction so that solutions obtained in section 6 will be employed.

8. FREQUENCIES OF A SINGLE-LAYERED SPHERICAL SHELL

8.1. FREQUENCY EQUATION OF THE FIRST CLASS ($n \geqslant 1$)

For a single-layered spherical shell, we should only allow for the boundary conditions (62), from which we obtain the frequency equation as follows:

$$|E_{ij}^1| = 0 \quad (i, j = 1, 2), \tag{66}$$

where

$$\begin{aligned}
E_{11}^1 &= [\eta - (3 + \alpha)/2]\,\mathrm{J}_\eta(\Omega\xi_1) - \Omega\xi_1\mathrm{J}_{\eta+1}(\Omega\xi_1),\\
E_{12}^1 &= [\eta - (3 + \alpha)/2]\,\mathrm{Y}_\eta(\Omega\xi_1) - \Omega\xi_1\mathrm{Y}_{\eta+1}(\Omega\xi_1),\\
E_{21}^1 &= [\eta - (3 + \alpha)/2]\,\mathrm{J}_\eta(\Omega\xi_2) - \Omega\xi_2\mathrm{J}_{\eta+1}(\Omega\xi_2),\\
E_{22}^1 &= [\eta - (3 + \alpha)/2]\,\mathrm{Y}_\eta(\Omega\xi_2) - \Omega\xi_2\mathrm{Y}_{\eta+1}(\Omega\xi_2).
\end{aligned} \tag{67}$$

Notice that when $n = 1$, the frequency equation (66) corresponds to a torsional or rotary mode of the shell. In particular, there exists a rigid-body rotation, for which the frequency equals zero. We can further see that equation (66) contains no parameter related to the electric field. In fact, it is identical to that of a pure elastic inhomogeneous spherical shell. For the homogeneous case, equation (66) degenerates to that reported in Ding and Chen [34, 35].

It is interesting to consider the case when $\Omega\xi_i$ $(i = 1, 2)$ are large and the spherical shell is thin, for which the asymptotic expansions of Bessel functions can be used [37]. We therefore derive the following frequency equation:

$$\frac{\tan(\Omega t^*)}{\Omega t^*} = \frac{4\eta^2 + 15 + 4\alpha}{8\Omega^2\xi_1\xi_2 - 4\eta^2 + 33 + 16\alpha + 2\alpha^2}, \tag{68}$$

where $t^* = (b - a)/R$ is the thickness-to-mean radius ratio of the shell. Equation (68) is expectedly identical to that obtained by Cohen *et al.* [30] if the inhomogeneity is not considered, i.e., when $\alpha = 0$.

8.2. FREQUENCY EQUATION OF THE SECOND CLASS ($n \geqslant 0$)

For $n = 0$, it needs further investigations. First, we denote the roots of the indicial equation (56) as follows:

$$s_{1,2} = \pm\sqrt{\frac{f_4(1+\alpha)^2 - 8(f_3 - f_1 - f_2 + \alpha f_3) + f_8[4f_6 - (1+\alpha)]^2}{4(f_4 + f_8)}}, \quad s_{3,4} = \mp(1+\alpha)/2. \tag{69}$$

We can then verify that the solution corresponding to s_4 will give no contribution to the stresses and electric displacements so that we have $C_{04} = 0$ in equation (57). That is to say only three unknown constants are involved for $n = 0$. It seems paradoxical at a glance that we have totally four boundary conditions ($\sigma_{rr} = D_r = 0$) on the inner and outer spherical surfaces. However, one can demonstrate that the radial electric displacement component D_r corresponding to ether s_1 or s_2 will be zero (see Appendix B), which guarantees that the two boundary conditions $D_r(a) = 0$ and $D_r(b) = 0$, both giving $C_{03} = 0$, are exactly the same. We thus finally obtain two linear homogeneous equations in two unknowns C_{01} and C_{02}. The vanishing of the corresponding coefficient determinant gives the following frequency equation:

$$|E^2_{ij}| = 0 \quad (i, j = 1, 2), \tag{70}$$

where

$$\begin{aligned} E^2_{1i} &= 2W_{0i}(\xi_1)/\xi_1 + (f_4/f_3)W'_{0i}(\xi_1) + (f_8/f_3)X'_{0i}(\xi_1) \\ E^2_{2i} &= 2W_{0i}(\xi_2)/\xi_2 + (f_4/f_3)W'_{0i}(\xi_2) + (f_8/f_3)X'_{0i}(\xi_2) \end{aligned} \quad (i = 1, 2). \tag{71}$$

Obviously, frequency equation (70) corresponds to the purely radial vibration. It is also noted here that if a multi-layered spherical shell is considered, only $2p$ unknowns will be involved in the final $2p$ linear homogeneous system.

When $n \geqslant 1$, one obtains six homogeneous linear algebraic equations and the following frequency equation is derived:

$$|E^3_{ij}| = 0 \quad (i, j = 1, 2, \ldots, 6), \tag{72}$$

where

$$E_{1i}^3 = n(n+1)V_{ni}(\xi_1)/\xi_1 + 2W_{ni}(\xi_1)/\xi_1 + (f_4/f_3)W'_{ni}(\xi_1) + (f_8/f_3)X'_{ni}(\xi_1)$$

$$E_{21}^3 = W_{ni}(\xi_1)/\xi_1 + V_{ni}(\xi_1)/\xi_1 - V'_{ni}(\xi_1) + f_5 f_8 X_{ni}(\xi_1)/\xi_1$$

$$E_{3i}^3 = n(n+1)f_6 V_{ni}(\xi_1)/\xi_1 + 2f_6 W_{ni}(\xi_1)/\xi_1 + W'_{ni}(\xi_1) - X'_{ni}(\xi_1) \qquad (i = 1, 2, \ldots, 6). \quad (73)$$

$$E_{4i}^3 = n(n+1)V_{ni}(\xi_2)/\xi_2 + 2W_{ni}(\xi_2)/\xi_2 + (f_4/f_3)W'_{ni}(\xi_2) + (f_8/f_3)X'_{ni}(\xi_2)$$

$$E_{5i}^3 = W_{ni}(\xi_2)/\xi_2 + V_{ni}(\xi_2)/\xi_2 - V'_{ni}(\xi_2) + f_5 f_8 X_{ni}(\xi_2)/\xi_2$$

$$E_{6i}^3 = n(n+1)f_6 V_{ni}(\xi_2)/\xi_2 + 2f_6 W_{ni}(\xi_2)/\xi_2 + W'_{ni}(\xi_2) - X'_{ni}(\xi_2)$$

It is noted that for $n = 1$, equation (72) has trivial solution, i.e., $\Omega = 0$. It corresponds to a rigid-body translation of the shell. The existence of such a rigid-body movement also has been observed for a homogeneous elastic spherical shell [29, 34].

So far, we have derived frequency equations of the two independent classes of vibrations. In the next section, we will give some numerical results to study the effects of several involved parameters of the spherical shell. We will pay attention only to the natural frequencies and the corresponding mode shapes are not to be presented. One can solve the eigenvectors from the homogeneous linear algebraic equations once the frequency is obtained and then can calculate the corresponding mode shapes by simple substitution.

9. NUMERICAL EXAMPLES

For numerical calculations, we shall consider two kinds of piezoelectric materials, i.e., PZT-4 and PZT-7A, whose material constants can be found in Dunn and Taya [38], for example. Table 1 lists out their non-dimensional values calculated according to the definitions given in equation (40). We use MATHEMATICA to write program and perform all the calculations. Since each frequency equation has more than one root, only the smallest positive frequency that is of physical significance will be presented in the following.

9.1. THE FIRST CLASS

As mentioned earlier, equation (66) is the frequency equation of torsional vibration when $n = 1$. Figure 2 displays curves of the non-dimensional torsional frequency Ω versus the inhomogeneity parameter α, for four values of the thickness-to-mean radius ratio, t^*. The material is taken to be PZT-4. It is seen that the thicker the shell is, the smaller the frequency

TABLE 1

Non-dimensional material constants of two piezoelectric materials

Material	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
PZT-4	5·43	3·04	2·90	4·49	0·84	− 0·34	1·15	1·58
PZT-7A	5·08	3·00	2·92	5·16	1·02	− 0·22	1·96	1·71

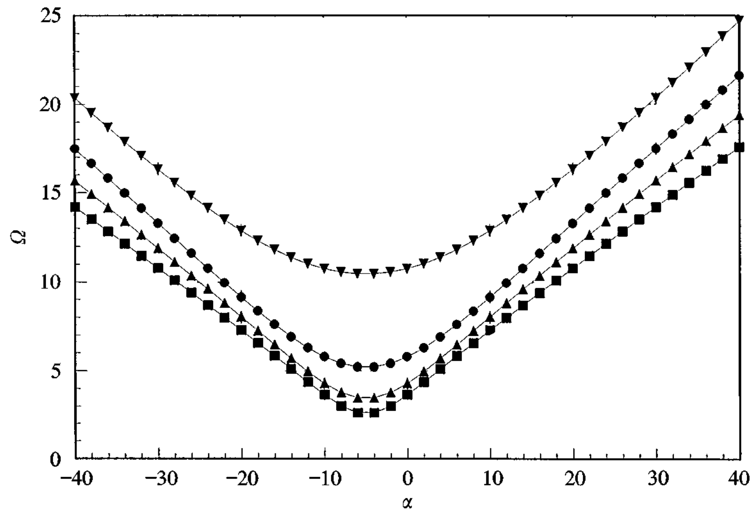

Figure 2. Non-dimensional torsional frequency Ω (first kind, $n = 1$) versus the inhomogeneity parameter α (PZT-4): —■—, $t^* = 1{\cdot}2$; —▲—, $t^* = 0{\cdot}9$; —●—, $t^* = 0{\cdot}6$; —▼—, $t^* = 0{\cdot}3$.

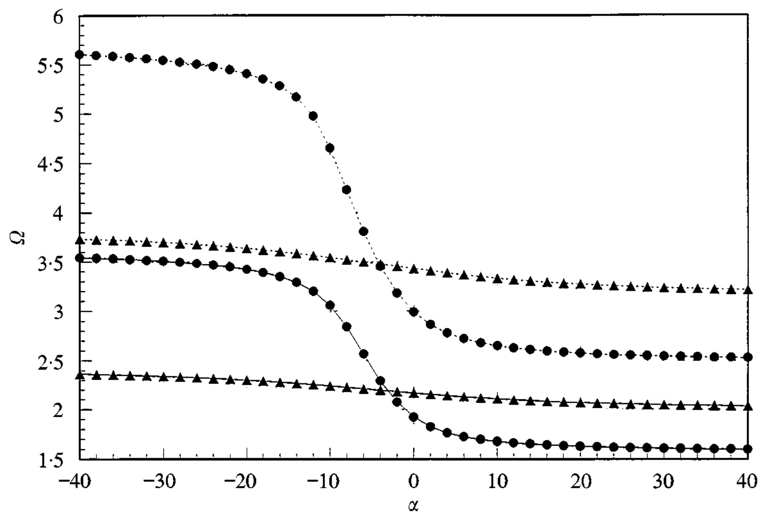

Figure 3. Non-dimensional frequency Ω of the first kind (solid line: $n = 2$; dotted line: $n = 3$) versus the inhomogeneity parameter α (PZT-4): —▲—, $t^* = 0{\cdot}2$; —●—, $t^* = 0{\cdot}8$; ····▲····, $t^* = 0{\cdot}2$; ···●···, $t^* = 0{\cdot}8$.

is. Also, with the increase of α, the frequency Ω first decreases and then becomes larger. The lowest natural frequency appears at $\alpha \approx -5{\cdot}0$. This will be of particular importance in practice since engineers can design the spherical dynamic gauges and meters not only by adopting different sizes but also by utilizing the idea of inhomogeneity. Figure 3 displays curves of the non-dimensional frequency Ω versus the inhomogeneity parameter α for $n = 2$ and 3. We notice the specified phenomena that occupied by the inhomogeneous spherical shell, i.e. for a certain value of α, the frequency curves corresponding to two different ratios of the thickness-to-mean radius t^* will intersect. By comparing Figure 2 with Figure 3, we find that, in contrast to the torsional vibration, the frequency for higher modes ($n = 2, 3$) decreases with the increase of α, and the degree of the descendant varies greatly with the value of t^*. Moreover, the effect of α on a thick shell is more obvious than that on a thin shell.

9.2. THE SECOND CLASS

For the second class, when $n = 0$, the spherical shell will vibrate only in the radial direction. This state of vibration is also known as the "breathing mode". Curves of the non-dimensional frequency Ω versus the inhomogeneity parameter α for three values of t^* are shown in Figures 4–6 respectively. For comparison purpose, we give two curves

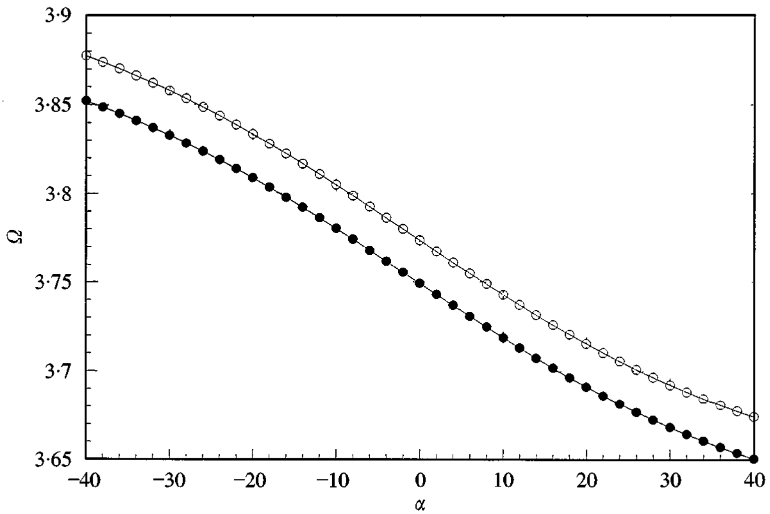

Figure 4. Non-dimensional breathing mode frequency Ω (second kind, $n = 0$) versus the inhomogeneity parameter α when $t^* = 0{\cdot}1$: $n = 0$; $t^* = 0{\cdot}1$; —○—, PZT-7A; —●—, PZT-4.

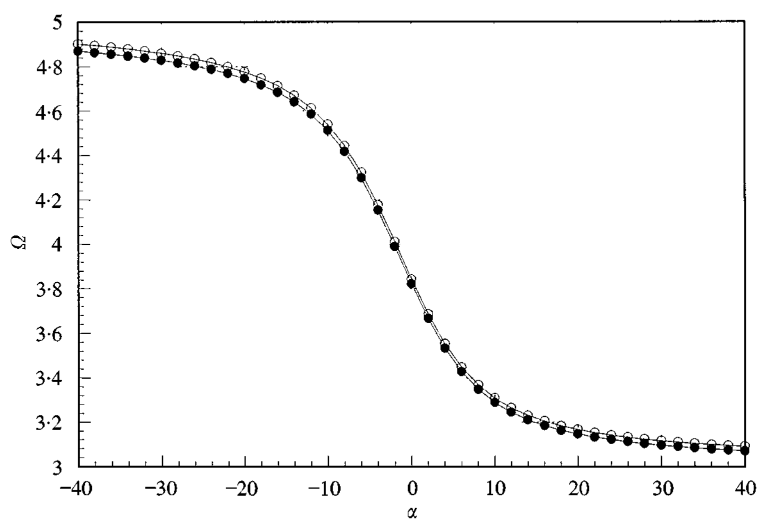

Figure 5. Non-dimensional breathing mode frequency Ω (second kind, $n = 0$) versus the inhomogeneity parameter α when $t^* = 0{\cdot}5$: $n = 0$; $t^* = 0{\cdot}5$; —○—, PZT-7A; —●—, PZT-4.

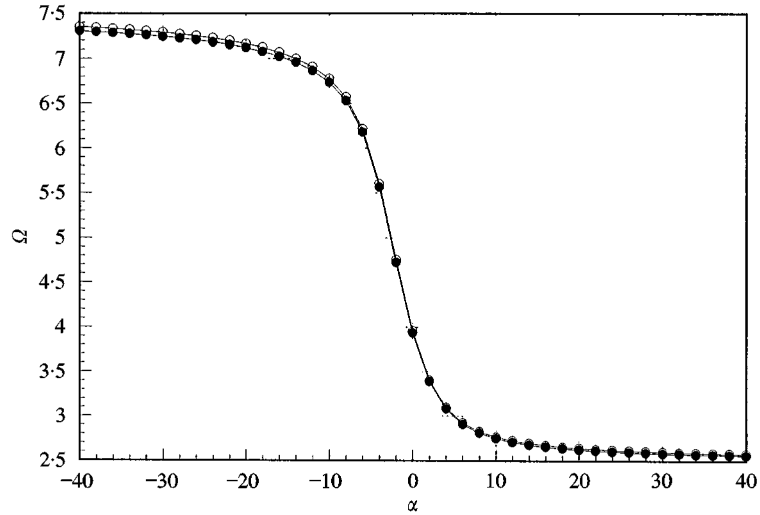

Figure 6. Non-dimensional breathing mode frequency Ω (second kind, $n = 0$) versus the inhomogeneity parameter α when $t^* = 1{\cdot}0$: $n = 0$; $t^* = 1{\cdot}0$; —○—, PZT-7A; —●—, PZT-4.

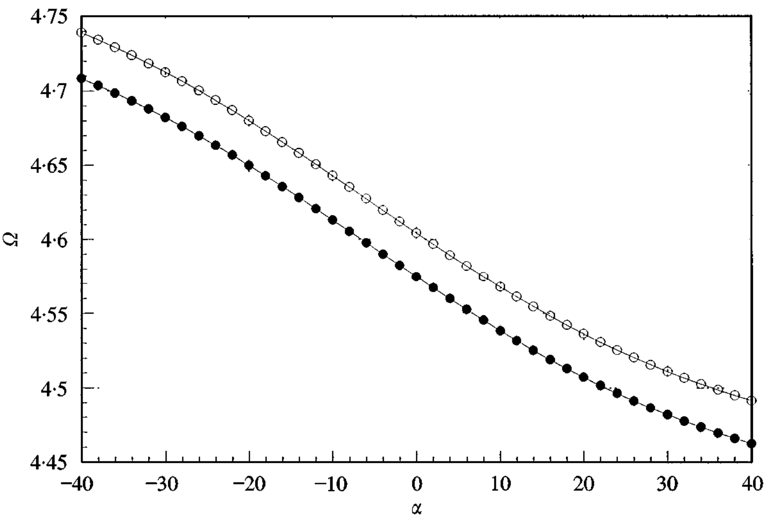

Figure 7. Non-dimensional frequency Ω (second kind, $n = 1$) versus the inhomogeneity parameter α when $t^* = 0{\cdot}1$: $n = 1$; $t^* = 0{\cdot}1$; —○—, PZT-7A; —●—, PZT-4.

corresponding to PZT-4 and PZT-7A materials in each figure simultaneously. It is shown that the frequency of a PZT-4 spherical shell is lower than the corresponding one of a PZT-7A spherical shell for all three values of the thickness-to-mean radius ratio. As we can see, the non-dimensional frequency Ω decreases with the increase of α for all cases when $n = 0$. Similar behaviors can be observed for the non-breathing mode when $n = 2$ as shown in Figures 10–12. However, for $n = 1$ (Figures 7–9), it is somehow different for the thicker shells as we can see from Figure 9, for which the thickness-to-mean radius ratio is 1·0. The variation of Ω versus α is no longer monotonous, see the region $0 < \alpha < 10$.

Though numerical results are not given for higher modes, other materials as well as other geometric parameters, the author believes that some particular observations will be

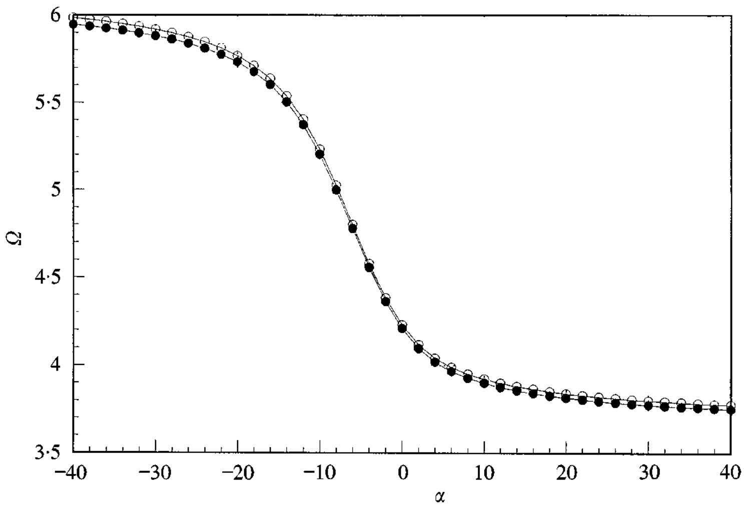

Figure 8. Non-dimensional frequency Ω (second kind, $n = 1$) versus the inhomogeneity parameter α when $t^* = 0{\cdot}5$: $n = 1$; $t^* = 0{\cdot}5$; —○—, PZT-7A; —●—, PZT-4.

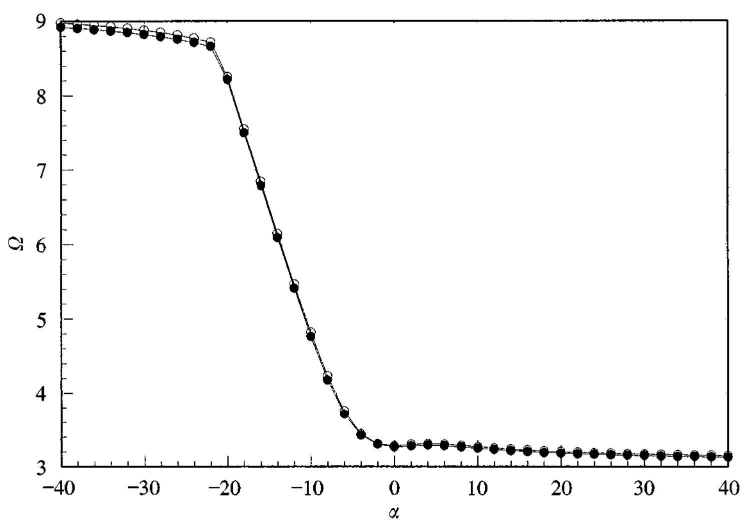

Figure 9. Non-dimensional frequency Ω (second kind, $n = 1$) versus the inhomogeneity parameter α when $t^* = 1{\cdot}0$: $n = 1$; $t^* = 1{\cdot}0$; —○—, PZT-7A; —●—, PZT-4.

obtained. It is impossible to present here a much wider numerical investigation on various parameters involved in the frequency equations due to the length limitation of the paper. In practice, one can perform such calculations without any difficulty and then extract some useful conclusions that will be helpful to a certain practical design.

10. CONCLUSIONS

In this paper, we simplify the basic equations of a radially polarized, non-homogeneous piezoelastic medium by the introduction of three displacement functions. For the general

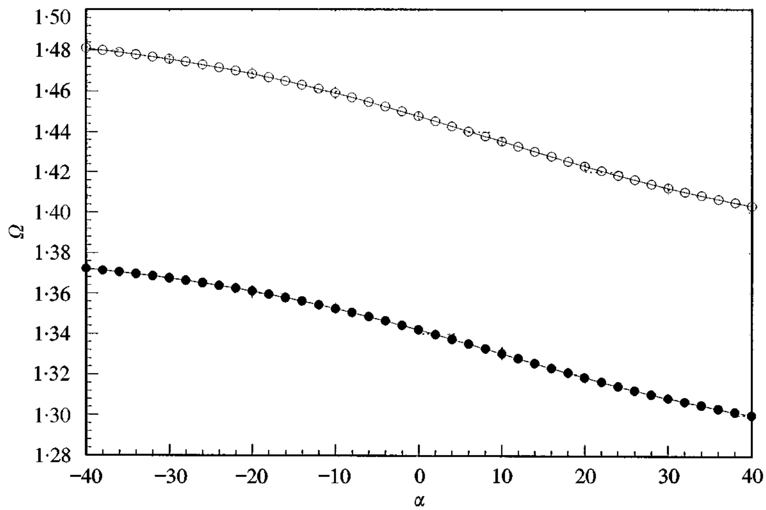

Figure 10. Non-dimensional frequency Ω (second kind, $n = 2$) versus the inhomogeneity parameter α when $t^* = 0{\cdot}1$: $n = 2$; $t^* = 0{\cdot}1$; —○—, PZT-7A; —●—, PZT-4.

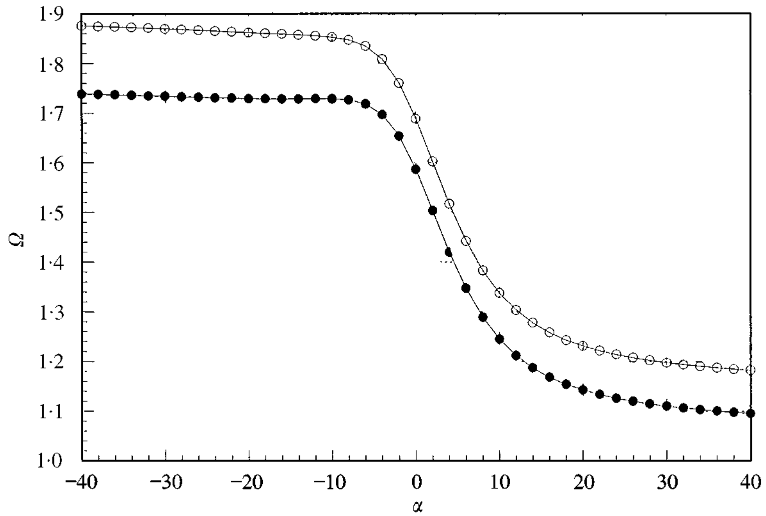

Figure 11. Non-dimensional frequency Ω (second kind, $n = 2$) versus the inhomogeneity parameter α when $t^* = 0{\cdot}5$: $n = 2$; $t^* = 0{\cdot}5$; —○—, PZT-7A; —●—, PZT-4.

non-axisymmetric free vibration problem, these equations are finally reduced to an uncoupled second order ordinary differential equation in the unknown U_n, and a coupled system of three such equations in the unknowns V_n, W_n and X_n. Generally speaking, solutions to these equations can always be obtained by the series expansion method. For the particular case that all the material constant are of power functions in the radial variable, the independent differential equation is found to be a special case of the confluent hypergeometric differential one and its solution is easily obtained. The coupled system is much complicated and its solution can be derived based on the matrix FPS method.

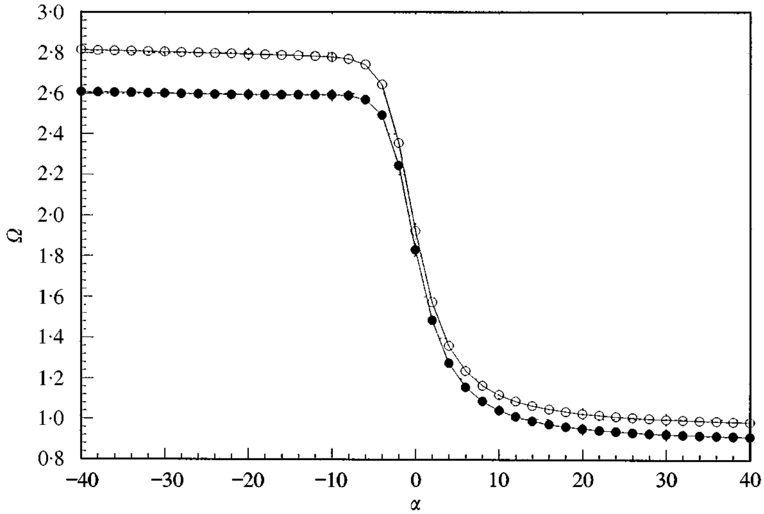

Figure 12. Non-dimensional frequency Ω (second kind, $n = 2$) versus the inhomogeneity parameter α when $t^* = 1{\cdot}0$: $n = 2$; $t^* = 1{\cdot}0$; —○—, PZT-7A; —●—, PZT-4.

The free vibration problem of multi-layered piezoelectric spherical shells is then considered. It is found that the vibration can be divided into two independent classes, just as the case of pure elasticity [29, 30]. In fact, the first class is identical to that of the pure elastic spherical shell, with no electric parameter involved. As expected, the second one has changed due to the specified coupling characteristics between the elastic and electric fields. We give for example the frequency equations of a single-layered piezoelastic spherical shell with material constants in power laws along the radial co-ordinate. In particular, we find that for the purely radial free vibration (the so-called "breathing mode"), the boundary conditions eventually include only two unknown constants and thus the frequency equation is simplified as equation (70) shows. This fact has not been found in the literature even for a single-layered homogeneous piezoelastic spherical shell. Numerical results are finally presented and some observations are obtained.

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APPENDIX A

In the following, we prove that to solve relative problems, one can assume $H \equiv 0$ so that equations (16) and (17) hold. It is first noticed that the resolvent form for displacements represented by equation (5) is not single. In fact, the corresponding homogeneous one of equation (5) has non-trivial solution namely ψ^0 and G^0 that satisfy:

$$\frac{1}{\sin\theta}\frac{\partial\psi^0}{\partial\phi}+\frac{\partial G^0}{\partial\theta}=0, \quad \frac{\partial\psi^0}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial G^0}{\partial\phi}=0. \tag{A1}$$

It is obvious that if ψ and G in equation (5) are replaced with $\psi^* = \psi + \psi^0$ and $G^* = G + G^0$, respectively, the mechanical displacement remains unaltered. From equation (A1), one finds that

$$\nabla_1^2\psi^0 = 0, \quad \nabla_1^2 G^0 = 0. \tag{A2}$$

Substituting ψ and G with ψ^* and G^*, respectively, into equations (13) and (14) will lead to equations (16) and (17) if there exist

$$[c_{44}\nabla_3^2 - 2(c_{44}-c_{66}) + c_{66}\nabla_1^2 + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\partial^2/\partial t^2]\psi^0 = \sin\theta\frac{\partial H}{\partial\theta}, \tag{A3}$$

$$[c_{44}\nabla_3^2 - 2(c_{44}-c_{66}) + c_{11}\nabla_1^2 + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\partial^2/\partial t^2]G^0 = -\frac{\partial H}{\partial\phi}. \tag{A4}$$

Noticing equations (A2), the above two equations turn to

$$[c_{44}\nabla_3^2 - 2(c_{44}-c_{66}) + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\partial^2/\partial t^2]\psi^0 = \sin\theta\frac{\partial H}{\partial\theta}, \tag{A5}$$

$$[c_{44}\nabla_3^2 - 2(c_{44}-c_{66}) + (\nabla_2 c_{44})(\nabla_2 - 1) - r^2\rho\partial^2/\partial t^2]G^0 = -\frac{\partial H}{\partial\phi}. \tag{A6}$$

Now consider the following equation:

$$\left[\nabla_3^2 - \frac{2(c_{44}-c_{66})}{c_{44}} + \frac{\nabla_2 c_{44}}{c_{44}}(\nabla_2 - 1) - \frac{r^2\rho}{c_{44}}\frac{\partial^2}{\partial t^2}\right]\Theta = H(r,t,\theta,\phi), \tag{A7}$$

where H is an arbitrary function. Applying the Laplace transform to equation (A7) gives

$$r^2\bar{\Theta}'' + r\left(2 + \frac{\nabla_2 c_{44}}{c_{44}}\right)\bar{\Theta}' - \frac{2(c_{44} - c_{66}) + \nabla_2 c_{44} + r^2 p^2 \rho}{c_{44}}\bar{\Theta} = \bar{H}(r, p, \theta, \phi), \quad \text{(A8)}$$

where

$$\bar{\Theta} = \int_0^\infty \Theta e^{-pt}\,dt \quad \bar{H} = \int_0^\infty H e^{-pt}\,dt.$$

It is seen that equation (A8) is a non-homogeneous, second order ordinary differential equation with certain singular points. Theoretically speaking, one can at first get the corresponding homogeneous solution by the theory of second order ordinary equation. Then one can derive the non-homogeneous solution by the method of variation of parameter. Assuming that the solution to equation (A8) has been obtained, we take

$$\bar{\psi}^0 = \sin\theta \frac{1}{c_{44}}\frac{\partial\bar{\Theta}}{\partial\theta}, \quad \bar{G}_0 = -\frac{1}{c_{44}}\frac{\partial\bar{\Theta}}{\partial\phi}, \quad \text{(A9)}$$

where $\bar{\psi}^0$ and $\bar{G}^0$ are the image functions of ψ^0 and G^0 respectively. Since $\bar{\Theta}$ is obtained by the method of variation of parameter, its expression includes the kernel $\bar{H}$. Moreover, it is known that both the Laplace transform as well as the solving procedure do not involve the two variables θ and ϕ. Thus, from equation (15), one has $\nabla_1^2\bar{\Theta} = 0$. Noticing that the elastic constant c_{44} is a function of the variable r only, it is obtained from equations (A9) that

$$\frac{1}{\sin\theta}\frac{\partial\bar{\psi}^0}{\partial\phi} + \frac{\partial\bar{G}^0}{\partial\theta} = 0, \quad \frac{\partial\bar{\psi}^0}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\bar{G}^0}{\partial\phi} = 0. \quad \text{(A10)}$$

Because the Laplace transform is only related to the time variable t, we can directly derive equation (A1) from the above equation. Based on the above verification, we obviously obtain equations (16) and (17).

APPENDIX B

In this appendix, we prove that the electric displacement D_r corresponding to the two indicials s_1 and s_2 vanishes when $n = 0$. According to the matrix FPS method [28], we assume that

$$\begin{Bmatrix} W_0 \\ X_0 \end{Bmatrix} = \sum_{i=0}^{\infty} \begin{Bmatrix} \bar{W}_0 \\ \bar{X}_0 \end{Bmatrix}_i \xi^{-(1+\alpha)/2+s+i}. \quad \text{(B1)}$$

For the sake of simplicity, we assume here that no two of the four indicials s_i $(i = 1, 2, 3, 4)$ differ by an even integer. Otherwise, the solution as given in equation (B1) may contain logarithmic terms [28] and the followed demonstration is similar to what will be presented here. Substituting equation (B1) into equations (54) and (55) gives

$$\sum_{i=0}^{\infty} [\mathbf{N}_1\xi^2 + \mathbf{N}_2(s+i)] \begin{Bmatrix} \bar{W}_0 \\ \bar{X}_0 \end{Bmatrix}_i \xi^{s+i} = 0, \quad \text{(B2)}$$

where

$$\mathbf{N}_1 = \begin{bmatrix} \Omega^2/f_4 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{N}_2(s+i) = \begin{bmatrix} si^2 - \dfrac{(1+\alpha)^2}{4} + \dfrac{2(f_3 - f_1 - f_2 + \alpha f_3)}{f_4} & \dfrac{f_8}{f_4}\left[si^2 - \dfrac{(1+\alpha)^2}{4} - 2f_6\left(si - \dfrac{1+\alpha}{2}\right)\right] \\ -\left[si^2 - \dfrac{(1+\alpha)^2}{4} + 2f_6\left(si + \dfrac{1+\alpha}{2}\right)\right] & si^2 - \dfrac{(1+\alpha)^2}{4} \end{bmatrix} \tag{B3}$$

and $si = s + i$. The indicial equation (56) can be obtained by setting $|\mathbf{N}_2(s)| = 0$. By virtue of the matrix FPS method described in reference [28], we obtain from equation (B2)

$$\begin{Bmatrix} W_0 \\ X_0 \end{Bmatrix} = \sum_{i=0}^{\infty} \begin{Bmatrix} \bar{W}_0 \\ \bar{X}_0 \end{Bmatrix}_{2i} \xi^{-(1+\alpha)/2+s+2i}. \tag{B4}$$

where

$$\begin{Bmatrix} W_0 \\ X_0 \end{Bmatrix}_0 = C_{00} \begin{Bmatrix} s - \dfrac{1+\alpha}{2} \\ s - \dfrac{1+\alpha}{2} + 2f_6 \end{Bmatrix}, \tag{B5}$$

$$\begin{Bmatrix} \bar{W}_0 \\ \bar{X}_0 \end{Bmatrix}_{2i} = \mathbf{N}_2(s+2i)^{-1}\mathbf{N}_1 \begin{Bmatrix} \bar{W}_0 \\ \bar{X}_0 \end{Bmatrix}_{2(i-1)} \quad (i = 1, 2, 3, \dots), \tag{B6}$$

in which C_{00} is an arbitrary constant. Noticing the properties of the two matrices $\mathbf{N}_1$ and $\mathbf{N}_2$, one can easily find that the following identity holds for $i \geqslant 1$:

$$\begin{Bmatrix} \bar{W}_0 \\ \bar{X}_0 \end{Bmatrix}_{2i} = \begin{Bmatrix} k_1 \bar{W}_0 \\ k_2 \bar{W}_0 \end{Bmatrix}_0, \tag{B7}$$

where k_1 and k_2 are two constants and the ratio between them can be obtained as

$$\frac{k_1}{k_2} = \frac{s + 2i - (1+\alpha)/2}{s + 2i + (1+\alpha)/2 + 2f_6}. \tag{B8}$$

From equations (58) and (B4), one obtains the expression for D_r when $n = 0$:

$$\begin{aligned} D_r/e_{33} &= 2f_6 W_0/\xi + W_0' - X_0' \\ &= \sum_{i=0}^{\infty}\left[\left(2f_6 + s + 2i - \frac{1+\alpha}{2}\right)\bar{W}_{0(2i)} - \left(s + 2i - \frac{1+\alpha}{2}\right)\bar{X}_{0(2i)}\right]\xi^{-(1+\alpha)/2+s+2i-1}. \end{aligned} \tag{B9}$$

By virtue of equations (B5), (B7) and (B8), we directly obtain $D_r = 0$ from equation (B9).

It should be pointed out that the above verification is only valid for the two indicials s_1 and s_2 ($s_4 = (1+\alpha)/2$ is no longer considered because the corresponding stresses and electric displacements vanish and thus can be made zero). For $s_3 = -(1+\alpha)/2$, since

$$\mathbf{N}_2(s) = \begin{bmatrix} \dfrac{2(f_3 - f_1 - f_2 + \alpha f_3)}{f_4} & \dfrac{2 f_8 f_6 (1+\alpha)}{f_4} \\ 0 & 0 \end{bmatrix}, \tag{B10}$$

we can only take

$$\begin{Bmatrix} \bar{W}_0 \\ \bar{X}_0 \end{Bmatrix}_0 = C_{00} \begin{Bmatrix} f_8 f_6 (1+\alpha) \\ f_1 + f_2 - (1+\alpha) f_3 \end{Bmatrix}. \tag{B11}$$

Thus, the radial electric displacement D_r corresponding to s_3 does not equal zero.